Loan Guarantees
Part VI - The Revised BSOPM - PDE and Proof

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In this white paper we will derive the PDE and prove that the equation for an uncapped guarantee is consistent with the PDE and boundary conditions. We will leave the proof that the capped guarantee is consistent with the PDE and boundary conditions to the ambitious reader.

In previous parts of this series on loan guarantees we determined that the equation for the value of an uncapped guarantee at time t was...

$$G_t = D_T\, f(\alpha,t)\, CND\left[d_1\right] - \Gamma A_t\, g(\phi,t)\, CND\left[d_2\right] \text{ ...where...}$$
$$d_1 = \left[\ln\left(\frac{D_T}{A_t}\right) - \left(\alpha - \phi - \frac{1}{2}\sigma^2\right)(T-t)\right] \Big/ \sigma\sqrt{T-t} \text{ ...and... } d_2 = d_1 - \sigma\sqrt{T-t} \tag{1}$$

Our Hypothetical Problem

We are tasked with calculating the price of a put option given the following assumptions...

Table 1: Model Assumptions

Symbol	Description	Value
A_0	Enterprise value at time zero	1,366,700
D_T	Debt payoff amount at time T	500,000
T	Guarantee term in years	3.0000
Γ	Liquidation value factor	0.5308
α	Risk-free rate (continuous time)	0.0392
ϕ	Dividend yield (continuous time)	0.0732
σ	Annual return volatility	0.3858

We will use our model to answer the following questions:

Question 1: Prove that Equation (1) above is consistent with the requisite boundary conditions.
Question 2: Prove that Equation (1) above is consistent with the PDE derived below.

Equations For Enterprise Value

We will define the variable A_t to be enterprise value at time t, the variable κ to be the continuous time cost of capital, the variable ϕ to be the continuous time dividend yield, and the variable δW_t to be the change in the driving Brownian motion, which is the source of all risk. The stochastic differential equation that defines the change in enterprise value over time is...

$$\delta A_t = (\kappa - \phi)\, A_t\, \delta t + \sigma\, A_t\, \delta W_t \text{ ...where... } \delta W_t \sim N\left[0, \delta t\right] \tag{2}$$

The solution to Equation (2) above is the equation for enterprise value at time t, which is...

$$A_t = A_0\, \text{Exp}\left\{(\kappa - \phi)\, t + \sigma\sqrt{t}\, Z\right\} \text{ ...where... } Z \sim \left[0, 1\right] \tag{3}$$

Note the following product rules...

$$\delta t^2 = 0 \ \text{...and...}\ \delta W_t^2 = \delta t \ \text{...and...}\ \delta t\, \delta W_t = 0 \tag{4}$$

Using Equations (2) and (4) above the equation for the square of the change in enterprise value over time is...

$$\begin{aligned}\delta A_t^2 &= (\kappa-\phi)^2 A_t^2 \delta t^2 + 2\,(\kappa-\phi)\,\sigma\,A_t^2\,\delta t\,\delta W_t + \sigma^2 A_t^2 \delta W_t^2 \\ &= (\kappa-\phi)^2 A_t^2 \times 0 + 2\,(\kappa-\phi)\,\sigma\,A_t^2 \times 0 + \sigma^2 A_t^2 \times \delta t \\ &= \sigma^2 A_t^2\,\delta t\end{aligned} \tag{5}$$

Equations For Guarantee Value

We will model guarantee value to be a function of time (t) and enterprise value (A_t). The general form of the equation for guarantee value at time t is...

$$G_t = f(G_t, t) \tag{6}$$

Given that guarantee value is once differentiable with respect to time and twice differentiable with respect to enterprise value, using Equation (6) above the equation for the change in guarantee value at time t is...

$$\delta G_t = \frac{\delta G_t}{\delta t}\,\delta t + \frac{\delta G_t}{\delta A_t}\,\delta A_t + \frac{1}{2}\,\frac{\delta^2 G_t}{\delta A_t^2}\,\delta A_t^2 \tag{7}$$

Once the hedge portfolio is set up all risk from being long or short the guarantee will be removed. Once risk is removed then all discounting will be done at the risk-free rate. We will define the function F_G to be the present value at time zero of the change in guarantee value at time t. Using the product rule the equation for F_G is...

$$F_G = \delta\Bigg[\text{Exp}\Bigg\{-\alpha\,t\Bigg\}G_t\Bigg] = \delta\Bigg[\text{Exp}\Bigg\{-\alpha\,t\Bigg\}\Bigg]G_t + \delta\Bigg[G_t\Bigg]\,\text{Exp}\Bigg\{-\alpha\,t\Bigg\} \tag{8}$$

Using Equation (7) above the solution to Equation (8) above is...

$$F_G = -\alpha\,\text{Exp}\Bigg\{-\alpha\,t\Bigg\}G_t\,\delta t + \text{Exp}\Bigg\{-\alpha\,t\Bigg\}\Bigg(\frac{\delta G_t}{\delta t}\,\delta t + \frac{\delta G_t}{\delta A_t}\,\delta A_t + \frac{1}{2}\,\frac{\delta^2 G_t}{\delta A_t^2}\,\delta A_t^2\Bigg) \tag{9}$$

Using Equation (2) and (5) above we can rewrite Equation (9) above as...

$$F_G = -\alpha\,\text{Exp}\Bigg\{-\alpha\,t\Bigg\}G_t\,\delta t + \text{Exp}\Bigg\{-\alpha\,t\Bigg\}\Bigg(\frac{\delta G_t}{\delta t}\,\delta t + \frac{\delta G_t}{\delta A_t}\Bigg((\kappa-\phi)\,A_t\,\delta t + \sigma\,A_t\,\delta W_t\Bigg) + \frac{1}{2}\,\frac{\delta^2 G_t}{\delta A_t^2}\,\delta t\Bigg) \tag{10}$$

After combining terms we can rewrite Equation (10) above as...

$$F_G = \text{Exp}\Bigg\{-\alpha\,t\Bigg\}\Bigg(\Bigg(-\alpha\,G_t + \frac{\delta G_t}{\delta t} + \frac{\delta G_t}{\delta A_t}\,(\kappa-\phi)\,A_t + \frac{1}{2}\,\frac{\delta^2 G_t}{\delta A_t^2}\,\sigma^2 A_t^2\Bigg)\delta t + \frac{\delta G_t}{\delta S_t}\,\sigma\,A_t\,\delta W_t\Bigg) \tag{11}$$

Equation For Hedge Portfolio Value

The hedge portfolio is comprised of a long/(short) position in enterprise value and a long/(short) position in a risk-free bond. We will define the variable P_t to be hedge portfolio value at time t and the variable Δ_t to be the percent of total enterprise value that is either long or short at time t. The equation for hedge portfolio value at time t is...

$$P_t = \Delta_t\,A_t + (P_t - \Delta_t\,A_t) \tag{12}$$

The return on the hedge portfolio consists of capital gains/(losses) on the long/(short) position in enterprise value, dividends on the long/(short) position in enterprise value, and the risk-free bond earning the risk-free rate. Using Equation (12) above the equation for the change in hedge portfolio value at time t is...

$$\delta P_t = \Delta_t\,\delta A_t + \alpha(P_t - \Delta_t\,A_t)\,\delta t + \phi\,\Delta_t\,A_t\,\delta t \tag{13}$$

We will define the function F_P to be the present value at time zero of the change in hedge portfolio value at time t. Using the product rule the equation for F_P is...

$$F_P = \delta\Bigg[\text{Exp}\Bigg\{-\alpha\,t\Bigg\}P_t\Bigg] = \delta\Bigg[\text{Exp}\Bigg\{-\alpha\,t\Bigg\}\Bigg]P_t + \delta\Bigg[P_t\Bigg]\,\text{Exp}\Bigg\{-\alpha\,t\Bigg\} \tag{14}$$

Using Equations (12) and (13) above the solution to Equation (14) above is...

$$F_P = -\alpha \, \mathrm{Exp}\bigg\{-\alpha \, t\bigg\} P_t \, \delta t + \mathrm{Exp}\bigg\{-\alpha \, t\bigg\}\bigg(\Delta_t \, \delta A_t + \alpha (P_t - \Delta_t \, A_t) \, \delta t + \phi \, \Delta_t \, A_t \, \delta t\bigg) \tag{15}$$

Using Equation (2) above we can rewrite Equation (15) above as...

$$F_P = -\alpha \, \mathrm{Exp}\bigg\{-\alpha \, t\bigg\} P_t \, \delta t + \mathrm{Exp}\bigg\{-\alpha \, t\bigg\}\bigg(\Delta_t \bigg((\kappa - \phi) \, A_t \, \delta t + \sigma A_t \, \delta W_t\bigg) + \alpha (P_t - \Delta_t \, A_t) \, \delta t + \phi \, \Delta_t \, A_t \, \delta t\bigg) \tag{16}$$

After combining terms we can rewrite Equation (16) above as...

$$F_P = \mathrm{Exp}\bigg\{-\alpha \, t\bigg\}\bigg(\Delta_t \, (\kappa - \alpha) \, A_t \, \delta t + \Delta_t \, \sigma A_t \, \delta W_t\bigg) \tag{17}$$

Build The Hedge

If we are short the guarantee then using Equations (8) and (14) above our short position risk is fully hedged given that the following equation holds at any time t...

$$F_G = F_P \tag{18}$$

If we multiply both sides of Equation (18) above by the discounting factor then that equation becomes...

$$\mathrm{Exp}\bigg\{\alpha \, t\bigg\} \, F_G = \mathrm{Exp}\bigg\{\alpha \, t\bigg\} \, F_P \tag{19}$$

Using Equations (11) and (17) above we can rewrite Equation (19) above as...

$$\Delta_t \, (\kappa - \alpha) \, A_t \, \delta t + \Delta_t \, \sigma A_t \, \delta W_t = \bigg(-\alpha \, G_t + \frac{\delta G_t}{\delta t} + \frac{\delta G_t}{\delta A_t} \, (\kappa - \phi) \, A_t + \frac{1}{2} \, \frac{\delta^2 G_t}{\delta A_t^2} \, \sigma^2 A_t^2\bigg) \delta t + \frac{\delta G_t}{\delta S_t} \, \sigma \, A_t \, \delta W_t \tag{20}$$

All of the risk of a short position in the guarantee is contained in the value of the Brownian motion term W_t. To remove all risk we want to get rid of the δW_t term in Equation (20) above so we will make the following definition...

$$\Delta_t = \frac{\delta G_t}{\delta A_t} \tag{21}$$

Using Equation (21) above we can rewrite Equation (20) above as...

$$\begin{aligned}
\frac{\delta G_t}{\delta A_t} \, (\kappa - \alpha) \, A_t \, \delta t + \frac{\delta G_t}{\delta A_t} \, \sigma \, A_t \, \delta W_t &= \bigg(-\alpha \, G_t + \frac{\delta G_t}{\delta t} + \frac{\delta G_t}{\delta A_t} \, (\kappa - \phi) \, A_t + \frac{1}{2} \, \frac{\delta^2 G_t}{\delta A_t^2} \, \sigma^2 A_t^2\bigg) \delta t + \frac{\delta G_t}{\delta A_t} \, \sigma \, A_t \, \delta W_t \\
\frac{\delta G_t}{\delta A_t} \, (\kappa - \alpha) \, A_t \, \delta t &= \bigg(-\alpha \, G_t + \frac{\delta G_t}{\delta t} + \frac{\delta G_t}{\delta A_t} \, (\kappa - \phi) \, A_t + \frac{1}{2} \, \frac{\delta^2 G_t}{\delta A_t^2} \, \sigma^2 A_t^2\bigg) \delta t \\
\frac{\delta G_t}{\delta A_t} \, \kappa \, A_t - \frac{\delta G_t}{\delta A_t} \, \alpha \, A_t &= -\alpha \, G_t + \frac{\delta G_t}{\delta t} + \frac{\delta G_t}{\delta A_t} \, \kappa \, A_t - \frac{\delta G_t}{\delta A_t} \, \phi + \frac{1}{2} \, \frac{\delta^2 G_t}{\delta A_t^2} \, \sigma^2 A_t^2 \\
0 &= -\alpha \, G_t + \frac{\delta G_t}{\delta t} + \frac{\delta G_t}{\delta A_t} \, (\alpha - \phi) \, A_t + \frac{1}{2} \, \frac{\delta^2 G_t}{\delta A_t^2} \, \sigma^2 A_t^2 \qquad (22)
\end{aligned}$$

Note that Equation (22) above is the PDE for a guarantee.

Selected Greeks

The equation for the Greek Delta, which is the first derivative of guarantee value with respect to enterprise value, from Part V is...

$$\frac{\delta G_t}{\delta A_t} = (\Gamma - 1) \, g(\phi, t) \, \frac{1}{\sigma \sqrt{T - t}} \, \frac{\delta CND[d_2]}{\delta d_2} - \Gamma g(\phi, t) \, CND[d_2] \tag{23}$$

The equation for the Greek Gamma, which is the second derivative of guarantee value with respect to enterprise value, from Part V is...

$$\frac{\delta^2 G_t}{\delta A_t^2} = (1 - \Gamma) \, g(\phi, t) \, \frac{A_t}{\sigma \sqrt{T - t}} \, \frac{\delta^2 CND[d_2]}{\delta d_2^2} \, \frac{\delta^2 d_2}{\delta A_t^2} - \Gamma g(\phi, t) \, \frac{\delta CND[d_2]}{\delta d_2} \, \frac{\delta d_2}{\delta A_t} \tag{24}$$

The equation for the Greek Theta, which is the first derivative of guarantee value with respect to time t, from Part V is...

$$\frac{\delta G_t}{\delta t} = D_T\, f(\alpha,t)\left(\alpha\, CND\left[d_1\right]+(1-\Gamma)\,\frac{\delta CND[d_1]}{\delta d_1}\,\frac{\delta d_1}{\delta t}\right)-\Gamma A_t\, g(\phi,t)\left(\phi\, CND\left[d_2\right]+\frac{\delta CND[d_2]}{\delta d_2}\,\frac{\sigma}{2\sqrt{T-t}}\right) \tag{25}$$

The Answers To Our Hypothetical Problem

Question 1: Prove that Equation (1) above is consistent with the requisite boundary conditions.

We determined that the value of an uncapped guarantee at time T given the value of the enterprise at time T is...

$$G_T = \text{Max}\left(D_T - \Gamma A_T, 0\right) \tag{26}$$

Since we set the values of d_1 and d_2 in Equation (1) above such that we integrate over the range where the guarantee is in-the-money we can drop the MAX in Equation (26) such that the equation for the value of an uncapped guarantee at time T given the value of the enterprise at time T is...

$$G_T = D_T - \Gamma A_T \tag{27}$$

Note that at time T Equation (1) above becomes...

$$G_t = D_T\, f(\alpha,t)\, CND\left[d_1\right] - \Gamma A_t\, g(\phi,t)\, CND\left[d_2\right] \text{ ...becomes... } G_T = D_T - \Gamma A_T \tag{28}$$

Since Equations (27) and (28) above are the same we proved that Equation (1) above is consistent with the requisite boundary conditions set out in Equation (26) above.

Question 2: Prove that Equation (1) above is consistent with the PDE derived above.

Assuming that enterprise value at the end of year one is \$1 million and at the end of year two is \$0.3 million, and using derivative equations (23), (24) and (25) above, the proof of the PDE via Equation (22) above, and the answer to Question 2, is...

Description	Year 0	Year 1	Year 2
Enterprise value	1,366,700	1,000,000	300,000
Guarantee value	41,869	52,667	323,173
PDE Proof:			
$-\alpha \times G_t$	-1,642.14	-2,065.65	-12,675.09
Theta	-21,949.57	-32,909.50	15,741.14
Delta $\times (\alpha - \phi) \times A_t$	3,423.90	4,891.69	6,649.35
Gamma $\times \frac{1}{2} \times \sigma^2 \times A_t^2$	20,167.82	30,083.46	-9,715.40
Total (should be = 0)	0.00	0.00	0.00